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Solution of Confined Vortex Problems[1]

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ABSTRACT

A numerical procedure is presented for the solution of the steady-state Navier–Stokes equations for the flow of a viscous, incompressible fluid between two rotating cylinders, with or without flow through the cylinders in the radial direction. The velocity calculated by this method agrees with experimental measurements, and the iterative process may be successfully accelerated to speed convergence in some cases and decelerated to prevent divergence in others.

INTRODUCTION

While attempting to determine the forces that cause and maintain such atmospheric phenomena as tornadoes and dust whirls, some investigators (e.g., Long [*1*]) have examined vortices confined in circular cylinders. In one such study, Pao [*2*] calculated the flow of a viscous, incompressible fluid confined in a cylinder where the top plate was rotating with constant angular velocity, the bottom plate was held stationary, and the cylinder was either rotating with the top plate or held fixed.

The use of a confined vortex was also an integral part of a conceptual nuclear reactor design investigated by Kerrebrock and Keyes [*3*] and Kerrebrock and Meghreblian [*4*]. The vortex, generated by tangentially injecting a gas of low molecular weight into a cylinder and exhausting it through a hole in one of the end walls, was used to maintain an annular cloud of fissioning material. In an experimental study related to this application, Kidd [*5*] examined the flow generated by passing a fluid radially inward through a pair of rotating, concentric, porous cylinders. One end plate of the cylinders was held stationary to simulate the condi-

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tions near the end plate of the vortex reactor, while both of the concentric cylinders and the other end plate rotated with a constant angular velocity.

In the past, such theoretical approaches as boundary layer approximations, momentum integral techniques and similarity transformations have been used to describe the interaction of a vortex with a stationary surface (see Lewellen [6], Schwiderski and Lugt [7], King [8], and Kidd and Farris [9]). However, the availability of large, high-speed, digital computers and the greater understanding of the use of numerical methods for solving nonlinear partial differential equations have made it practical to investigate flow problems without making the approximations inherent in these techniques.

In this paper, a numerical method is presented that was used to study two flow problems related to tornadoes and the reactor design discussed above. Consider a pair of concentric circular cylinders that are rotating with the same constant angular velocity Ω. The top end wall of the cylinder system is held stationary, while the other end wall also rotates with angular velocity Ω. In one of the problems the steady-state flow of an incompressible fluid contained in the cylinder system, with no flow across any portion of the boundary, is studied. In the other problem, there is uniform flow of the fluid radially inward through the cylinder boundaries.

In the remainder of this paper, the governing equations with boundary conditions appropriate to the two problems are presented and nondimensional equations in terms of the classical stream function and vorticity are derived. A numerical procedure for solving finite difference approximations of the latter equations is provided, and the results of numerical experiments carried out with this method are exhibited and discussed.

Governing Equations

It is assumed that the flow patterns in the two problems are axisymmetric and that the effect due to forces external to the system (such as the gravitational force) are negligible. Then, in terms of the radial, tangential and axial components (u, v, and w, respectively) of the fluid velocity and the pressure P, the equations of motion (in cylindrical coordinates) that result from conservation of mass and momentum are

$$\frac{1}{r}(ru)_r + w_z = 0, \tag{1}$$

$$uu_r - \frac{1}{r^3}\Gamma^2 + wu_z = -\frac{1}{\rho}P_r + \nu\left[\left(\frac{1}{r}(ru)_r\right)_r + u_{zz}\right], \tag{2}$$

$$u\Gamma_r + w\Gamma_z = \nu\left[r\left(\frac{1}{r}\Gamma_r\right)_r + \Gamma_{zz}\right], \tag{3}$$

$$uw_r + ww_z = -\frac{1}{\rho}P_z + \nu\left[\frac{1}{r}(rw_r)_r + w_{zz}\right], \tag{4}$$

where the subscripts denote partial differentiation, ρ and ν are the constant mass density and kinematic viscosity of the fluid, and the function Γ is defined by

$$\Gamma(r, z) \equiv rv(r, z). \tag{5}$$

For the problem with no fluid flow through the boundary of the system, the boundary conditions are

$$u(\underline{r}, z) = w(\underline{r}, z) = 0, \qquad \Gamma(\underline{r}, z) = \underline{r}^2\Omega, \tag{6}$$

$$u(\bar{r}, z) = w(\bar{r}, z) = 0, \qquad \Gamma(\bar{r}, z) = \bar{r}^2\Omega, \tag{7}$$

for $0 \leqslant z \leqslant \bar{z}$, and

$$u(r, 0) = w(r, 0) = 0, \qquad \Gamma(r, 0) = r^2\Omega, \tag{8}$$

$$u(r, \bar{z}) = \Gamma(r, \bar{z}) = w(r, \bar{z}) = 0, \tag{9}$$

for $\underline{r} \leqslant r \leqslant \bar{r}$, where $\underline{r}$ and $\bar{r}$ are the radii of the inner and outer cylinders and $\bar{z}$ is the height of the cylinder system. When there is flow through the boundary of the system, the conditions (6)–(9) at the boundary are valid, except that

$$u(\underline{r}, z) = \nu R'/\underline{r}, \tag{6'}$$

$$u(\bar{r}, z) = \nu R'/\bar{r}, \tag{7'}$$

for $0 \leqslant z \leqslant \bar{z}$, where R' is the radial Reynolds number.

The pressure may be eliminated from (2) and (4) by differentiating (2) with respect to z and (4) with respect to r and by subtracting the results, so that the motion can be described by three equations in terms of u, Γ, and w. These equations may be nondimensionalized by dividing the independent variables by a characteristic length L, which we take to be $\bar{r}$, and by dividing the velocity terms by $L\Omega$. In terms of these nondimensional quantities, we define the stream function $\psi(r, z)$ by

$$u \equiv \frac{1}{r}\psi_z, \qquad w \equiv -\frac{1}{r}\psi_r, \tag{10}$$

and the vorticity $\zeta(r, z)$ by

$$\zeta(r, z) \equiv u_z - w_r. \tag{11}$$

It is readily verified that ψ, as defined in (10), identically satisfies the nondimensional formulation of (1) and that (2)–(4) can be written as

$$\Delta\Gamma - \frac{1}{r}\Gamma_r = R\left(\frac{1}{r}\psi_z\Gamma_r - \frac{1}{r}\psi_r\Gamma_z\right), \tag{12}$$

$$\Delta\zeta + \frac{1}{r}\zeta_r - \frac{1}{r^2}\zeta = R\left(\frac{1}{r}\psi_z\zeta_r - \frac{1}{r}\psi_r\zeta_z - \frac{1}{r^2}\zeta\psi_z - \frac{2}{r^3}\Gamma\Gamma_z\right), \tag{13}$$

$$\Delta\psi - \frac{1}{r}\psi_r = r\zeta, \tag{14}$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2},$$

and the tangential Reynolds number R is defined by

$$R \equiv L^2\Omega/\nu.$$

For the details of the derivation see Farris *et al.* [10].

In terms of these variables, (6)–(9) can be written as

$$\Gamma(\underline{r}, z) = \underline{r}^2, \qquad \zeta(\underline{r}, z) = \frac{1}{\underline{r}}\,\psi_{rr}(\underline{r}, z), \qquad \psi(\underline{r}, z) = 0 \tag{15}$$

$$\Gamma(\bar{r}, z) = \bar{r}^2, \qquad \zeta(\bar{r}, z) = \frac{1}{\bar{r}}\,\psi_{rr}(\bar{r}, z), \qquad \psi(\bar{r}, z) = 0, \tag{16}$$

for $0 \leqslant z \leqslant \bar{z}$, and

$$\Gamma(r, 0) = r^2, \qquad \zeta(r, 0) = \frac{1}{r}\,\psi_{zz}(r, 0), \qquad \psi(r, 0) = 0, \tag{17}$$

$$\Gamma(r, \bar{z}) = \psi(r, \bar{z}) = 0, \qquad \zeta(r, \bar{z}) = \frac{1}{r}\,\psi_{zz}(r, \bar{z}), \tag{18}$$

for $\underline{r} \leqslant r \leqslant \bar{r}$, where the $\underline{r}$, $\bar{r}$, and $\bar{z}$ used in (6)–(9) have been nondimensionalized by the factor $1/L$. When there is a flow through the cylinder walls, (15)–(18) are valid, except that

$$\psi(\underline{r}, z) = \underline{r}zu(\underline{r}, z), \qquad \psi(\bar{r}, z) = \bar{r}zu(\bar{r}, z), \tag{15'}$$

for $0 \leqslant z \leqslant \bar{z}$, and

$$\psi(r, \bar{z}) = r\bar{z}u(r, \bar{z}), \tag{18'}$$

for $\underline{r} \leqslant r \leqslant \bar{r}$.

Numerical Method

As a result of the axial symmetry in both problems, only the flow in the rectangular region $D = \{(r, z) \mid \underline{r} \leqslant r \leqslant \bar{r} \text{ and } 0 \leqslant z \leqslant \bar{z}\}$ needed to be considered. On D a network of uniformly spaced grid lines was constructed and, at each interior mesh point (the intersection of two grid lines), the derivatives that appear in (12)–(14) were approximated by central differences. The resulting difference equations contain a truncation error of order h^2, where h is the grid size (see Smith

[11]). During the iterative process the difference equations were used to obtain temporary values for Γ, ζ, and ψ as follows:

$$\begin{aligned}\Gamma^*_{i,j} &= \frac{1}{4}(\Gamma_{i+1,j} + \Gamma_{i,j+1} + \Gamma^{(n+1)}_{i-1,j} + \Gamma^{(n+1)}_{i,j-1}) \\ &- \frac{R}{16r_i}\Big[\Big(\psi_{i,j+1} - \psi_{i,j-1} + \frac{2h}{R}\Big)(\Gamma_{i+1,j} - \Gamma^{(n+1)}_{i-1,j}) \\ &- (\psi_{i+1,j} - \psi_{i-1,j})(\Gamma_{i,j+1} - \Gamma^{(n+1)}_{i,j-1})\Big], \end{aligned} \tag{19}$$

$$\begin{aligned}\zeta^*_{i,j} &= \Big\{\frac{1}{4}(\zeta_{i+1,j} + \zeta_{i,j+1} + \zeta^{(n+1)}_{i-1,j} + \zeta^{(n+1)}_{i,j-1}) \\ &- \frac{R}{16r_i}\Big[\Big(\psi_{i,j+1} - \psi_{i,j-1} - \frac{2h}{R}\Big)(\zeta_{i+1,j} - \zeta^{(n+1)}_{i-1,j}) \\ &- (\psi_{i+1,j} - \psi_{i-1,j})(\zeta_{i,j+1} - \zeta^{(n+1)}_{i,j-1}) \\ &- \frac{4h}{r_i^2}\Gamma^{(n+1)}_{i,j}(\Gamma^{(n+1)}_{i,j+1} - \Gamma^{(n+1)}_{i,j-1})\Big]\Big\} \\ &\div \Big\{1 + \Big(\psi_{i,j-1} - \psi_{i,j+1} + \frac{2h}{R}\Big)\frac{hR}{8r_i^2}\Big\}, \end{aligned} \tag{20}$$

$$\begin{aligned}\psi^*_{i,j} &= \frac{1}{4}(\psi_{i+1,j} + \psi_{i,j+1} + \psi^{(n+1)}_{i-1,j} + \psi^{(n+1)}_{i,j-1}) \\ &- \frac{h}{8r_i}(2r_i^2 h\zeta^{(n+1)}_{i,j} + \psi_{i+1,j} - \psi^{(n+1)}_{i-1,j}), \end{aligned} \tag{21}$$

where dependent variables without superscripts are assumed to have the superscript (n). An asterisk denotes a temporary value, say $\Lambda^*_{i,j}$, for the new iterate $\Lambda^{(n+1)}_{i,j}$ at the point $(r_i, z_j) = (r + (i-1)h, (j-1)h)$. The new iterate was obtained from this temporary value and the old iterate $\Lambda^{(n)}_{i,j}$ by the relaxation procedure

$$\Lambda^{(n+1)}_{i,j} = (1 - \omega_\Lambda)\Lambda^{(n)}_{i,j} + \omega_\Lambda \Lambda^*_{i,j},$$

where

$$0 < \omega_\Lambda < 2.$$

For the two problems, the boundary values used for (19) and (21) were immediately available from (15)–(18), (15′) and (18′). The boundary values for (20) were obtained from the following difference approximations for the derivatives that appear in (15)–(18):

$$\zeta_{1,j}^{(n+1)} = \frac{2}{\underline{r}h^2}(\psi_{2,j}^{(n)} - \psi_{1,j}), \tag{22}$$

$$\zeta_{M,j}^{(n+1)} = \frac{2}{\bar{r}h^2}(\psi_{M-1,j}^{(n)} - \psi_{M,j}), \tag{23}$$

for $1 \leqslant j \leqslant N$, and

$$\zeta_{i,1}^{(n+1)} = \frac{2}{r_i h^2}(\psi_{i,2}^{(n)} - \psi_{i,1}), \tag{24}$$

$$\zeta_{i,N}^{(n+1)} = \frac{2}{r_i h^2}(\psi_{i,N-1}^{(n)} - \psi_{i,N}), \tag{25}$$

for $2 \leqslant i \leqslant M - 1$, where M and N are the numbers of grid lines in the r- and z-directions and were chosen so that

$$h = \frac{\bar{z}}{N-1} = \frac{\bar{r} - \underline{r}}{M-1}.$$

The iteration procedure consisted of making sweeps of the interior mesh points (from left to right, then from bottom to top) for each of the dependent variables in turn, in the order Γ, ζ, and ψ. This was continued until

$$\max_{\Lambda=\Gamma,\zeta,\psi} \max_{D_h} \left| \frac{\Lambda_{i,j}^{(n+1)} - \Lambda_{i,j}^{(n)}}{\omega_\Lambda \Lambda_{i,j}^{(n+1)}} \right| < \epsilon, \tag{26}$$

where D_h is the set of interior mesh points.

For each different case, specified by the ordered pair (R, R'), an initial guess was provided by the results for some convergent case (R_0, R_0'), where $R_0 \leqslant R$ and $|R_0'| \leqslant |R|$.

Results of Numerical Experiments

This numerical procedure was used to calculate "solutions" of the problems specified by (12)–(14) with boundary values given by (15)–(18), (15′) and (18′) for R in the range 0.25 to 300 when $R' = 0$, and for R in the range 121 to 260 when R' was in the range 0 to -13.7 (where the minus sign denotes flow in the negative r-direction). In an effort to insure that the values obtained from the iterative procedure did in fact satisfy (12)–(14), the residual (the amount by which the difference equation fails to be satisfied) was calculated at each point of D_h for Γ, ζ, and ψ once (26) had been satisfied for $\epsilon = 0.001$. That the difference equations were satisfied is shown by the fact that not only were the residuals small (usually five orders of magnitude smaller than the value of the dependent variable at a point), but they were extremely sensitive to changes in the dependent variables, i.e., small changes in the dependent variables resulted in relatively large changes in the residuals.

Once solutions of the difference equations had been obtained, the values of the velocity components were calculated from difference equations analogous to (5), (10), and (11). In experimental studies performed by Kidd [5] for the problem with radial flow, measurements were made of the ratio of the radial to tangential velocity component at points near the stationary end wall, and of the tangential velocity component at points far from that wall. The corresponding numerical results were in good agreement with these measurements.

For the problem without radial flow, insufficient experimental results were obtained for a thorough comparison with the numerical results. With regard to the overall flow pattern, however, both the experimental and numerical results indicate that, far from the end wall, a potential vortex (the tangential velocity component is inversely proportional to the radial distance and the radial and axial velocity components vanish) is approximated when there is radial flow and that nearly solid body rotation exists when there is no radial flow. The details of the comparison between the calculated and experimental velocity components and of the presentation and analysis of the flow patterns in these two problems are given by Farris *et al.* [10].

The numerical procedure described in the previous section is basically that proposed by Pao [2] except that he did not use relaxation, i.e., he took $\omega_\Lambda = 1$ for $\Lambda = \Gamma$, ζ, and ψ, and his convergence criterion involved only the stream function. Although there is little general theory for the use of relaxation in the solution of nonlinear partial differential equations, relaxation has been employed in certain flow problems (e.g., Tejeira [12]) to accelerate convergence for some Reynolds numbers and to obtain convergence for others at which divergence would otherwise occur.

In an attempt to determine what relation, if any, exists between the relaxation vector $\boldsymbol{\omega} = (\omega_\Gamma, \omega_\zeta, \omega_\psi)$ and the tangential Reynolds number R, and to determine whether, for given R, changes in $\boldsymbol{\omega}$ greatly affect the number of iterations required for convergence, several numerical experiments were performed for R in the range 70 to 200 when there is no radial flow, i.e., $R' = 0$.

For each R, the vector $\boldsymbol{\omega}$ that resulted in the fewest iterations required for the convergence of the solution is displayed in Table I. Also, the number of iterations required so that each dependent variable satisfies a criterion similar to (26) is given.

TABLE I

R	ω			Iteration		
	Γ	ζ	ψ	Γ	ζ	ψ
70	1.4	0.8	1.4	50	625	350
80	1.4	0.8	1.4	50	700	200
90	1.4	0.7	1.3	50	950	100
100	1.4	0.7	1.3	50	700	100
150	1.3	0.2	1.2	150	1075	350
200	1.1	0.1	1.0	150	2000	350

In each of these cases, the initial guess used was the solution for the preceding case and the initial guess for the case $R = 70$ was the solution at $R = 60$. As R increases, the relaxation factors must be decreased in order that the number of iterations required for convergence be minimized. The vorticity is the most sensitive variable with respect to changes in R.

In Tables II and III, the results obtained by varying ω for the cases $R = 80$ and $R = 200$ are exhibited. In the cases where the ζ iteration had not converged, an examination of the quantity

$$\bar{\zeta} = \max_{D_h} \left| \frac{\zeta^{(n+1)} - \zeta^{(n)}}{\omega_\zeta \zeta^{(n+1)}} \right|$$

at every 50 iterations indicated that convergence would likely have been obtained had sufficiently many iterations been allowed to take place.

TABLE II
$R = 80$

ω			Iteration		
Γ	ζ	ψ	Γ	ζ	ψ
1.45	1.45	1.45	50	1000[a]	600
1.4	1.4	1.4	50	1000[a]	400
1.3	1.3	1.3	50	1000[a]	400
1.2	1.2	1.2	50	1000[a]	450
1.1	1.1	1.1	50	1000[a]	450
1.2	1.0	1.3	50	1000[a]	400
1.4	1.0	1.4	50	900	300
1.4	0.9	1.4	50	800	200
1.4	0.8	1.4	50	700	200

[a] The dependent variable had not converged in the indicated number of iterations.

TABLE III

$R = 200$

ω			Iteration		
Γ	ζ	ψ	Γ	ζ	ψ
1.1	0.4	1.0	350	2000[a]	1050
1.1	0.2	1.0	150	2000[a]	600
1.1	0.1	1.0	150	2000	350
1.1	0.09	1.0	150	2000[a]	350
1.1	0.08	0.9	200	2000[a]	400

[a] The dependent variable had not converged in the indicated number of iterations.

The sensitivity and lack of convergence of the vorticity ζ are again very much in evidence. From a study of these tables, it appears that there is an "optimal" relaxation vector, and that ω_Γ and ω_ψ can be considerably larger than ω_ζ. The results provided in Tables II and III are typical of those obtained in similar experiments for the other values of R in Table I (see Farris *et al.* [*10*]).

The quality of the initial guess required for convergence is a matter of considerable practical importance, since the interesting cases in many flow problems are characterized by fairly high Reynolds numbers. Although a comprehensive study of the required quality was not made, several of the cases in Table I were rerun using the solution at an even smaller Reynolds number for the initial guess. For each such case, the number of iterations required for convergence was not substantially increased. This was probably due to the fact that the overall flow pattern underwent no drastic alteration over the range of R that was studied (see Farris *et al.* [*10*]).

For the problem with radial flow, some cases and the corresponding relaxation vectors that led to convergence are given in Table IV. In general, many more iterations at considerably smaller relaxation factors were required for convergence, and the iterative process was extremely sensitive to the quality of the initial guess. When the relaxation factors were somewhat larger than those at which convergence eventually occurred, divergence often resulted within a few iterations. The sensitivity to initial guess seemed reasonable since the flow pattern changes rapidly with changes in R'.

In view of the fast convergence of Γ and ψ relative to ζ, it seemed that once Γ and ψ converged, i.e., individually satisfied a criterion similar to (26), the iterative process could be stopped and that the ζ calculated directly from a difference analogue of (14) might satisfy (26). Were this the case, substantial reductions in computer time could have been achieved. However, this was tried in several cases and the ζ that resulted was in some cases an improvement over the values obtained

TABLE IV

R	R'	ω		
		Γ	ζ	ψ
121	0 to −4.0	1.0	0.1	1.0
121	−6.0	0.5	0.04	0.5
121	−8.0	0.3	0.02	0.3
121	−9.66	0.1	0.01	0.1
164	−9.99	0.15	0.005	0.15
168	−9.76	0.15	0.005	0.15
176.1	−10.23	0.15	0.005	0.15
227.5	−10.23	0.12	0.004	0.12
230	−13.7	0.05	0.001	0.05
260	−9.83	0.12	0.004	0.12

from the iterative process, but in every case it fell considerably short of convergence. There were some cases in which no improvement was obtained. It is possible that the overall iterative procedure could be accelerated by periodically using this method to calculate ζ directly.

Summary and Comments

A numerical procedure has been developed for the solution of the Navier–Stokes equations for the steady flow of a viscous, incompressible fluid between two rotating cylinders, with or without flow through the cylinders in the radial direction. The velocity components calculated by this method agree with experimental measurements, and the iterative process may be successfully accelerated to speed convergence in some cases and decelerated to prevent divergence in others. It appears that there is an upper limit for the practical application of the method in its present form for the problem with radial flow, because of the degree to which the procedure must be underrelaxed to prevent divergence and the resulting rather large amounts of computer time required for convergence. For the problem without radial flow, it also appears that there is a practical upper limit, but that the present calculations are well below it.

Since the velocity components are calculated from the difference analogues of (5) and (10) in terms of Γ and ψ only, it might seem that the time spent obtaining convergence for ζ once Γ and ψ had converged was wasted. This would certainly have been the case if the velocity components had been the only variables of interest and if there were no desire for the results for some case involving Reynolds numbers beyond a given case (R, R'). However, the latter was not true and, in order to obtain a good initial guess for later cases, a well-converged ζ was essential, especially for the problem with radial flow.

An alteration of this procedure, which may result in an extension of its applicability to cases involving Reynolds numbers beyond those now possible, consists of using noncentral difference approximations to the first order partial derivatives that appear in (12)–(14) rather than the central differences now employed. In one- and two-dimensional analogues of the Navier-Stokes equations, Burns [*13*] and Boughner [*14*] showed that the replacement of central by noncentral difference approximations for first order derivatives resulted in substantially increased rates of convergence for sufficiently large Reynolds numbers. However, when such a substitution was made in the case $R = 200$, $R' = 0$, much slower convergence resulted.

Two other methods that should be considered in an attack on this and similar problems are the generalized Newton's method employed by Greenspan [*15, 16*], which solves the equations in terms of the stream function and vorticity, and the method proposed by Chorin [*17, 18*], which solves the equations in terms of velocity components.

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